

FINITE-RANK BRATTOLI–VERSHIK HOMEOMORPHISMS ARE EXPANSIVE

TAKASHI SHIMOMURA

ABSTRACT. Downarowicz and Maass (2008) have shown that every Cantor minimal homeomorphism with finite topological rank $K > 1$ is expansive. Bezuglyi, Kwiatkowski, and Medynets (2009) extended the result to non-minimal aperiodic cases. In this paper, we show that all finite-rank zero-dimensional systems are expansive or have infinite odometer systems; this is an extension of the two aforementioned results. Nevertheless, the methods follow similar approaches.

1. INTRODUCTION

Herman, Putnam, and Skau [HPS] have shown that a zero-dimensional system is essentially minimal if and only if it is represented as the Bratteli–Vershik system of an essentially simple ordered Bratteli diagram (see Definition 2.5). In [AK, Proposition 2.2], Akin and Kolyada completely characterized proximal topological dynamical systems. According to them, a topological dynamical system (X, f) is proximal if and only if it is essentially minimal and the unique minimal set is a fixed point. Thus, zero-dimensional proximal systems have Bratteli–Vershik representations. In [D, M], Danilenko and Matui studied not only Cantor minimal systems but also locally compact Cantor minimal systems. Locally compact Cantor minimal systems become proximal by one-point compactification. Thus, it is necessary to study the quality of essentially minimal systems whose minimal sets are fixed points. In this paper, a *zero-dimensional system* implies a pair (X, f) of a compact zero-dimensional metrizable space X and a homeomorphism $f : X \rightarrow X$. *Odometer systems* are always infinite. In this paper, we show that every finite-rank homeomorphic Bratteli–Vershik system without odometer systems is symbolic. This is an elaborate task. For zero-dimensional minimal systems, Downarowicz and Maass [DM] presented a remarkable theorem that states that every zero-dimensional minimal system of finite topological rank $K > 1$ is expansive. They used properly ordered Bratteli diagrams and adopted a noteworthy technique. In [BKM], Bezuglyi, Kwiatkowski, and

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Medynets extended the aforementioned result to non-minimal cases that do not have periodic orbits. Further, we showed that every zero-dimensional system has non-trivial Bratteli–Vershik representations (see Theorem 2.7 or [S]). In this paper, we show that the symbolcity still holds for all finite-rank homeomorphic Bratteli–Vershik systems without odometers, in which periodic orbits may be allowed. Our main result is as follows: if a zero-dimensional system has finite topological rank and no odometers, then it is expansive (see Theorem 4.4). The design of the proof presented in this paper essentially follows [DM] and the observation by Bezuglyi, Kwiatkowski, and Medynets [BKM].

2. PRELIMINARIES

Let \mathbb{Z} denote the set of all integers, and let \mathbb{N} denote the set of all non-negative integers.

Definition 2.1. A *Bratteli diagram* is an infinite directed graph (V, E) , where V is the vertex set and E is the edge set. These sets are partitioned into non-empty disjoint finite sets $V = V_0 \cup V_1 \cup V_2 \cup \dots$ and $E = E_1 \cup E_2 \cup \dots$, where $V_0 = \{v_0\}$ is a one-point set. Each E_n is a set of edges from V_{n-1} to V_n . Therefore, there exist two maps $r, s : E \rightarrow V$ such that $r : E_n \rightarrow V_n$ and $s : E_n \rightarrow V_{n-1}$ for all $n \geq 1$, i.e., the *range map* and the *source map*, respectively. Moreover, $s^{-1}(v) \neq \emptyset$ for all $v \in V$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$. We say that $u \in V_{n-1}$ is connected to $v \in V_n$ if there exists an edge $e \in E_n$ such that $s(e) = u$ and $r(e) = v$. The *rank* K of a Bratteli diagram is defined as $K := \liminf_{n \rightarrow \infty} \#V_n$, where $\#V_n$ is the number of elements in V_n .

Let (V, E) be a Bratteli diagram and $m < n$ be non-negative integers. We define

$$E_{m,n} := \{p \mid p \text{ is a path from a } u \in V_m \text{ to a } v \in V_n\}.$$

For a path $p = (e_{m+1}, e_{m+2}, \dots, e_n) \in E_{m,n}$, we define the source map $s(p) := s(e_{m+1})$ and the range map $r(p) := r(e_n)$. Then, we can construct a new Bratteli diagram (V', E') as follows:

$$V' := V_0 \cup V_1 \cup \dots \cup V_m \cup V_n \cup V_{n+1} \cup \dots$$

$$E' := E_1 \cup E_2 \cup \dots \cup E_m \cup E_{m,n} \cup E_{n+1} \cup \dots$$

This procedure is called *telescoping*. For a path $p = (e_{m+1}, e_{m+2}, \dots, e_n) \in E_{m,n}$ and $m \leq a < b \leq n$, we denote $p|_{[a,b]} := (e_{a+1}, e_{a+2}, \dots, e_b)$. For $n \geq 0$, we denote $E_{n,\infty}$ as the set of all infinite paths from V_n . For a path $p \in E_{n,\infty}$ and $n \leq a < b < \infty$, we define $p|_{[a,b]}$ as before, and we also define $p|_{[a,\infty)}$. Let $0 \leq m < n \leq \infty$ and $m \leq a \leq n$. For $v \in V_a$, we denote

$E_{m,n}(v) := \{p \in E_{m,n} \mid p \text{ passes } v\}$. Let $p \in E_{m,n}$. We define a closed and open set $C(p) := \{x \in E_{0,\infty} \mid x|_{[m,n]} = p\}$, and call it a *cylinder*.

Definition 2.2. Let (V, E) be a Bratteli diagram such that $V = V_0 \cup V_1 \cup V_2 \cup \dots$ and $E = E_1 \cup E_2 \cup \dots$ are the partitions, where $V_0 = \{v_0\}$ is a one-point set. Let $r, s : E \rightarrow V$ be the range map and source map, respectively. We say that (V, E, \leq) is an *ordered* Bratteli diagram if the partial order \leq is defined on E such that $e, e' \in E$ are comparable if and only if $r(e) = r(e')$. In other words, we have a linear order on each set $r^{-1}(v)$ with $v \in V \setminus V_0$. The edges $r^{-1}(v)$ are numbered from 1 to $\#(r^{-1}(v))$. Because $r^{-1}(v)$ is linearly ordered, we denote the maximal edge $e(v, \max) \in r^{-1}(v)$ and the minimal edge $e(v, \min) \in r^{-1}(v)$.

Let $n > 0$ and $e = (e_n, e_{n+1}, e_{n+2}, \dots), e' = (e'_n, e'_{n+1}, e'_{n+2}, \dots)$ be cofinal paths from the vertices of V_{n-1} , which might be different. We obtain the lexicographic order $e < e'$ as follows:

if $k \geq n$ is the largest number such that $e_k \neq e'_k$, then $e_k < e'_k$.

Definition 2.3. Let (V, E, \leq) be an ordered Bratteli diagram. Let E_{\max} and E_{\min} denote the sets of maximal and minimal edges, respectively. An infinite (resp. finite) path is maximal (resp. minimal) if all the edges constituting the path are elements of E_{\max} (resp. E_{\min}).

Definition 2.4. For $0 \leq m < n < \infty$, we denote $E_{m,n,\max} := \{p \mid p \in E_{m,n} \text{ and if } p = (e_{m+1}, e_{m+2}, \dots, e_n), \text{ then } e_i \in E_{\max} \text{ for all } i (m+1 \leq i \leq n)\}$. We also denote $E_{m,n,\min}$ similarly. In the same manner, we denote $E_{m,\infty,\max}$ and $E_{m,\infty,\min}$.

Definition 2.5. As in [HPS], an ordered Bratteli diagram (V, E) is called *essentially simple* if the following exist: a unique infinite path $p_{\max} = (e_{\max,1}, e_{\max,2}, \dots)$ with $e_{\max,i} \in E_{\max} \cap E_i$ for all $i \geq 1$, and a unique infinite path $p_{\min} = (e_{\min,1}, e_{\min,2}, \dots)$ with $e_{\min,i} \in E_{\min} \cap E_i$ for all $i \geq 1$.

Definition 2.6 (Vershik map). Let (V, E, \leq) be an ordered Bratteli diagram. Let $E_{0,\infty}$ be endowed with the subspace topology of the product space $\prod_{i=1}^{\infty} E_i$, with the discrete topology on each E_i ($1 \leq i < \infty$). Suppose that there exists a bijective map $\phi : E_{0,\infty,\max} \rightarrow E_{0,\infty,\min}$. Then, we can define a map $\phi : E_{0,\infty} \rightarrow E_{0,\infty}$ as follows:

If $e = (e_1, e_2, \dots) \neq p_{\max}$, then there exists the least $n \geq 1$ such that e_n is not maximal in $r^{-1}(r(e_n))$. Then, we can select the least $f_n > e_n$ in $r^{-1}(r(e_n))$. Let $v_{n-1} = s(f_n)$. Then, it is easy to obtain the unique least path $(f_1, f_2, \dots, f_{n-1})$ from v_0 to v_{n-1} . We define

$$\phi(e) := (f_1, f_2, \dots, f_{n-1}, f_n, e_{n+1}, e_{n+2}, \dots).$$

By definition, the map ϕ is bijective. Suppose that the map $\phi : E_{0,\infty} \rightarrow E_{0,\infty}$ is a homeomorphism. Then, it is called a *Vershik map*. The system $(E_{0,\infty}, \phi)$ is a zero-dimensional system (see [HPS]), and it is called the *Bratteli–Vershik system* of (V, E, \leq) .

If a zero-dimensional system (X, f) is topologically conjugate to a Bratteli–Vershik system $(E_{0,\infty}, \phi)$, then it is called a Bratteli–Vershik representation of (X, f) . For each vertex $v \in V_n$ with $n \geq 1$, the set of cylinders $\{C(p) \mid p \in E_{0,n}, r(p) = v\}$ constitutes a *tower* that is denoted as $T(v) := \bigcup_{p \in E_{0,n}, r(p)=v} C(p)$. We showed the following in [S]:

Theorem 2.7. *Let (X, f) be a zero-dimensional system, and let $0 < l_1 < l_2 < \dots$ be an arbitrary infinite sequence of integers. Then, (X, f) has a Bratteli–Vershik representation with an ordered Bratteli diagram $(\{V_n\}_{n \geq 0}, \{E_n\}_{n \geq 1})$ such that, if $\#E_{0,n}(v) \leq l_n$ for some $v \in V_n$, then there exists a sequence $(v_n = v, v_{n+1}, v_{n+2}, \dots)$ of vertices $v_{n+i} \in V_{n+i}$ ($i \geq 0$) such that there exists an $e_{n+i+1} \in E_{n+i+1}$ with $\{e_{n+i+1}\} = r^{-1}(v_{n+i+1})$ and $s(e_{n+i+1}) = v_{n+i}$ for all $i \geq 0$.*

In [DM], Downarowicz and Maass introduced the topological rank for a Cantor minimal homeomorphism. We define the same for all zero-dimensional systems.

Definition 2.8. Let (X, f) be a zero-dimensional system. Then, the *topological rank* of (X, f) is $1 \leq K \leq \infty$ if it has a Bratteli–Vershik representation with an ordered Bratteli diagram of rank K , and K is the minimum of such numbers.

Question. In general, the topological rank in the context of all zero-dimensional systems is less than or equal to the original. However, for Cantor minimal systems, the two topological ranks might be expected to coincide.

3. SOME PREPARATIONS AND RELATED RESULTS

To prepare the proof of our main result, we essentially follow [DM] and the observation by Bezuglyi, Kwiatkowski, and Medynets [BKM]. Let (V, E, \geq) be an ordered Bratteli diagram. For each $n \geq 1$, we write $r_n := \#V_n$ and $V_n = \{v_{n,1}, v_{n,2}, \dots, v_{n,r_n}\}$. We fix an $e = (e_1, e_2, \dots) \in E_{0,\infty}$. For all $i \in \mathbb{Z}$, we denote $\phi^i(e) = (e_{1,i}, e_{2,i}, \dots)$. Further, we denote $u_{n,i} = r(e_{n,i}) \in V_n$ ($n \geq 1, i \in \mathbb{Z}$). For a vertex $v \in V_n$, let $l(v) := \#E_{0,n}(v)$. When $(e_{1,i}, e_{2,i}, \dots, e_{n,i})$ is minimal and $r(e_{n,i}) = v$, we change the symbol $u_{n,i} = v$ to $u_{n,i} = \check{v}$. We write as $x_e := (u_{n,i})_{n \geq 1, i \in \mathbb{Z}}$. We define a shift map $(\sigma(x_e))_{n,i} = (u_{n,i+1})$ ($n \geq 1, i \in \mathbb{Z}$). Let $\bar{V}_n := V_n \cup \{\check{v}_{n,1}, \check{v}_{n,2}, \dots, \check{v}_{n,r_n}\}$ with the discrete topology. We consider the shift map $\sigma : \prod_{n \geq 1} \bar{V}_n^{\mathbb{Z}} \rightarrow \prod_{n \geq 1} \bar{V}_n^{\mathbb{Z}}$ with the

$v_{n,1}$	$v_{n,3}$	$v_{n,1}$	$v_{n,3}$	$v_{n,2}$	$v_{n,1}$
$v_{n+1,5}$		$v_{n+1,1}$			$v_{n+1,3}$

FIGURE 1. n th and $(n + 1)$ th rows of a linked array system with cuts.

v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0
$c_{1,3}$	$v_{1,3}$			$v_{1,1}$			$v_{1,3}$			$v_{1,2}$		$v_{1,1}$		
$v_{2,3}$				$v_{2,1}$								$v_{2,3}$		
$c_{3,3}$				$v_{3,1}$										
\vdots														

FIGURE 2. The first 4 rows of an array system.

v_0	v_0	v_0	v_0	v_0	v_0	v_0	v_0
$v_{1,1}$			$v_{1,3}$			$v_{1,2}$	
$v_{2,1}$							

FIGURE 3. The 2-symbol corresponding to the vertex $v_{2,1}$ of Figure 2.

product topology. There exists a dynamical embedding $\psi : (E_{0,\infty}, \phi) \rightarrow (\prod_{n \geq 1} \bar{V}_n^{\mathbb{Z}}, \sigma)$ with $\psi(e) = x_e$. We write $(X, f) := \psi(E_{0,\infty}, \phi)$. Fix an $n \geq 1$ arbitrarily. Following [DM], instead of using \check{v} , we make a *cut* just before each occurrence of \check{v} (see Figure 1). This coincides with the *array system* (see [DM]). Therefore, for each $x \in X$ and $n \geq 1$, there exists a unique sequence $x[n] := (\dots, u_{n,-2}, u_{n,-1}, u_{n,0}, u_{n,1}, \dots)$ of vertices of V_n that is separated by the cuts. We make a convention $x[0] = (\dots, v_0, v_0, v_0, \dots)$ that is cut everywhere. We also write $x(n, i) := u_{n,i}$ for all $x \in X, n \geq 0$, and $i \in \mathbb{Z}$. For an interval $[n, m]$ with $m > n$, the combination of rows $x[n']$ with $n \leq n' \leq m$ is denoted as $x[n, m]$. The *array system* of x is the infinite combination $x[0, \infty)$ of all rows $x[n]$ ($0 \leq n < \infty$). Note that for $m > n$, if there exists an m -cut at position i (just before position i), then there exists an n -cut at position i (just before position i). For each vertex $v_{n,i}$, if we write

$\{e_1 < e_2 < \dots < e_{k(n,i)}\} = r^{-1}(v_{n,i})$, we can determine a series of vertices $v_{n-1,a(n,i,1)}v_{n-1,a(n,i,2)} \dots v_{n-1,a(n,i,k(n,i))}$ such that $v_{n-1,a(n,i,j)} = s(e_j)$ ($1 \leq j \leq k(n,i)$). Furthermore, each $v_{n-1,a(n,i,j)}$ determines a series of vertices of level $n - 2$ similarly. Thus, we can determine a set of vertices arranged in a square form as in Figure 3. Following [DM], this form is said to be the *n-symbol* and denoted by $v_{n,i}$. For $m < n$, the projection $v_{n,i}[m]$ that is a finite sequence of vertices of V_m is also defined. It is clear that $x[n] = x'[n]$ implies that $x[0, n] = x'[0, n]$. If $x \neq x'$ ($x, x' \in X$), then there exists an $n > 0$ with $x[n] \neq x'[n]$. For $x, x' \in X$, we say that the pair (x, x') is

n -compatible if $x[n] = x'[n]$. If $x[n] \neq x'[n]$, then we say that x and x' are n -separated. We recall that if there exists an n -cut at position k , then there exists an m -cut at position k for all $0 \leq m \leq n$. Let $x \neq x'$. If a pair (x, x') is n -compatible and $(n+1)$ -separated, then we say that the *depth of compatibility* of x and x' is n , or the pair (x, x') has *depth* n . If (x, x') is a pair of depth n and (x, x'') is a pair of depth $m > n$, then the pair (x', x'') has depth n (hence, never equal). An n -separated pair (x, x') is said to *have a common n -cut* if both x and x' have an n -cut at the same position. If a pair has a common n -cut, then it also has a common m -cut for all m ($0 \leq m \leq n$). The set $X_n := \{x[n] \mid x \in X\}$ is a two-sided subshift of a finite set \bar{V}_n . Just after the n -cuts, we have changed each symbol $v_{n,i}$ to $\tilde{v}_{n,i}$. Thus, X_n is a two-sided subshift of a finite set \bar{V}_n . The factoring map is denoted by $\pi_n : X \rightarrow X_n$, and the shift map is denoted by $\sigma_n : X_n \rightarrow X_n$. We simply write $\sigma = \sigma_n$ for all n if there is no confusion. Let (Y, g) be a subsystem of $(E_{0,\infty}, \phi)$, i.e., $Y \subset E_{0,\infty}$ is closed, $\phi(Y) = Y$, and $g = \phi|_Y$. Let $V' := \{v \in V \mid T(v) \cap Y \neq \emptyset\}$. We note that for all p, p' with $r(p) = r(p') \in V_n$ ($n \geq 1$), $C(p) \cap Y \neq \emptyset$ if and only if $C(p') \cap Y \neq \emptyset$. Thus, for each $n > 0$, $p \in E_{0,n}$ satisfies $C(p) \cap Y \neq \emptyset$ if and only if $r(p) \in V'$. Let $E' := \{e \in E \mid r(e) \in V'\}$. Then, it is easy to check that (V', E') is a Bratteli diagram. We can give (V', E') the order \geq induced from the original. All the maximal (resp. minimal) paths of (V', E', \geq) are maximal (resp. minimal) paths of (V, E, \geq) . Let Y_{\min} (Y_{\max}) be the set of all minimal (maximal) paths of (V', E', \geq) . Then, it is obvious that $\phi(Y_{\min}) = Y_{\max}$. Now, (V', E', \geq) is an ordered Bratteli diagram, and the Bratteli–Vershik system that is topologically conjugate to (Y, g) is defined. Therefore, we get the following:

Theorem 3.1. *Let (X, f) be a zero-dimensional system of finite topological rank K . Then, its subsystem has topological rank $\leq K$.*

The next proposition follows [BKM, Proposition 4.6]. Nevertheless, because the proposition was previously described in the context of aperiodic systems, we present the proof here as well. We fix a metric d on $E_{0,\infty}$. Let $M, M' \subseteq E_{0,\infty}$ be closed sets. Then, we denote

$$\text{dist}(M, M') := \min_{x \in M, y \in M'} d(x, y).$$

For a closed set $M \subseteq E_{0,\infty}$, we denote $\text{diam}(M) := \max_{x, y \in M} d(x, y)$.

Proposition 3.2. *Let (X, f) be a zero-dimensional system of finite rank K . Then, (X, f) has at most K minimal sets.*

Proof. Suppose that there exist $K+1$ minimal sets, namely M_1, M_2, \dots, M_{K+1} . Take $\varepsilon > 0$ such that $\varepsilon < \min_{1 \leq i < j \leq K+1} \text{dist}(M_i, M_j)$ and $n > 0$ such that

$\max\{\text{diam}(C(p)) \mid p \in E_{0,n}\} \leq \varepsilon$. For each i ($1 \leq i \leq K+1$), we define $P_i := \{p \in E_{0,n} \mid C(p) \cap M_i \neq \emptyset\}$. It is evident that $P_i \neq \emptyset$ for all i ($1 \leq i \leq K+1$). It is also obvious that $P_i \cap P_j = \emptyset$ for $i \neq j$. We note that for each $v \in V_n$, $\{C(p) \mid p \in E_{0,n}, r(p) = v\}$ is a tower that constructs $T(v)$. Thus, for each $v \in V_n$ and i ($1 \leq i \leq K+1$), only one of $\{p \in E_{0,n} \mid r(p) = v\} \subseteq P_i$ or $\{p \in E_{0,n} \mid r(p) = v\} \cap P_i = \emptyset$ occurs, which is a contradiction. \square

4. MAIN THEOREM.

Before the statement of our main theorem, we choose a small $\varepsilon_0 > 0$. Let $M_1, M_2, \dots, M_{K'}$ ($K' \leq K$) be the list of all the minimal sets of $(E_{0,\infty}, \phi)$. Fix $\varepsilon_0 > 0$ such that $\varepsilon_0 < \frac{1}{3} \min\{\text{dist}(M_i, M_j) \mid 1 \leq i < j \leq K'\}$. Suppose that there exist $x \neq y \in E_{0,\infty}$ such that $\limsup_{n \rightarrow +\infty} d(f^n(x), f^n(y)) \leq \varepsilon_0$. Then, it is easy to see that the minimal sets of $\omega(x)$ and $\omega(y)$ coincide. Thus, we get the next two lemmas:

Lemma 4.1. *Let $x, y \in E_{0,\infty}$. Then, we get the following:*

- *if $\limsup_{n \rightarrow +\infty} d(f^n(x), f^n(y)) \leq \varepsilon_0$, then the minimal sets of $\omega(x)$ and $\omega(y)$ coincide.*

Lemma 4.2. *Let $x, y \in E_{0,\infty}$. Then, we get the following:*

- *if $\limsup_{n \rightarrow -\infty} d(f^n(x), f^n(y)) \leq \varepsilon_0$, then the minimal sets of $\alpha(x)$ and $\alpha(y)$ coincide.*

From the two lemmas stated above, we get the next lemma:

Lemma 4.3. *Let $x, y \in E_{0,\infty}$. If $\sup_{n \in \mathbb{Z}} d(f^n(x), f^n(y)) \leq \varepsilon_0$, then the minimal sets of $\alpha(x)$ and $\alpha(y)$ coincide and the minimal sets of $\omega(x)$ and $\omega(y)$ coincide.*

Theorem 4.4 (Main Result). *Let (X, f) be a finite-rank zero-dimensional system such that no minimal set is an odometer. Then, (X, f) is expansive.*

Proof. Let $K \geq 1$ be the topological rank of (X, f) . We note that if $K = 1$, then we obtain an ordered Bratteli diagram that has rank $K = 1$. Then, because (X, f) is not an odometer, the Bratteli–Vershik system is a single periodic orbit and is expansive. Thus, the conclusion of the theorem is obvious. Therefore, we assume that $K > 1$. As with the proof presented in [DM], we show our proof by contradiction. Suppose that the claim fails. Then, for all $L > 0$, there exists a pair (x, x') with distinct elements of X that is L -compatible. Because $x \neq x'$, for some $m > L$, (x, x') is m -separated. Therefore, (x, x') has depth n with $L \leq n < m$. Therefore, for infinitely many n , there exists a pair (x_n, x'_n) of depth n . By telescoping,

we can assume that every $n > 0$ has a pair (x_n, x'_n) of depth n . Note that even after another telescoping, this quality still holds. As with the proof presented in [DM], we show our proof for separate cases:

- (1) there exists an N such that for all $n > N$ and every $m > n$, there exists a pair (x_n, x'_n) of depth n with a common m -cut;
- (2) for infinitely many n , and every sufficiently large $m > n$, any pair of depth n has no common m -cut.

Proof for case (1). In [BKM], the proof is omitted. However, we present it here for the readers' convenience. As with the proof presented in [DM], we prove that such a case never occurs. Fix some $m > N + K$, and for each integer $n \in [m - K, m - 1]$, let (x_n, x'_n) be a pair of depth n with a common m -cut. For $n = m - 1$, we have an $(m - 1)$ -compatible m -separated pair (x_{m-1}, x'_{m-1}) with a common m -cut. Suppose that all the m -cuts of x_{m-1} and x'_{m-1} are the same. Because the pair (x_{m-1}, x'_{m-1}) is m -separated, at least two distinct vertices are in the same place of $x_{m-1}[m]$ and $x'_{m-1}[m]$. These symbols have the same rows from 0 to $m - 1$. Therefore, at least two distinct m -symbols have the same rows from 0 to $m - 1$. If the sets of m -symbols have the same rows from 0 to $m - 1$, we factor these to the same alphabet, i.e., we make a new ordered Bratteli diagram identifying such vertices of V_m . Suppose that v and v' are identified to a single v . Then, we assume that the sets $r^{-1}(v)$ and $r^{-1}(v')$ are identified to a single $r^{-1}(v)$ with the same order. In addition, we assume that $s^{-1}(v)$ and $s^{-1}(v')$ are joined to form the new $s^{-1}(v)$, and the orders are not changed. Further, $E_{0,\infty}$ has a canonical isomorphism to the original, and the X_{m-1} of the new Bratteli diagram is the same as the old one. Because the pair (x_{m-2}, x'_{m-2}) was $(m - 1)$ -separated, and this factorization does not affect the $(m - 1)$ th row, the pair (x_{m-2}, x'_{m-2}) is still m -separated. Note that the number of vertices of the m th row has been decreased by at least 1. Next, we consider the case in which after a common m -cut of the pair (x_{m-1}, x'_{m-1}) , the coincidence of the positions of the m -cuts does not continue toward the right or left end. Suppose that after a common m -cut at position k_0 , the common m -cuts do not continue to the right end. Let $k_1 > k_0$ be the position of the first common m -cut such that the right m -symbols v and v' of x_{m-1} and x'_{m-1} have different lengths. We assume without loss of generality that $l(v') < l(v)$. We recall that for any vertex $v \in V \setminus V_0$, $l(v) := \#E_{0,n}(v)$. Let $k_2 = k_1 + l(v')$. Then, k_2 is the position of the next m -cut of x'_{m-1} . Because $x_{m-1}[m - 1] = x'_{m-1}[m - 1]$, the m -symbol v itself has an $(m - 1)$ -cut at $l(v')$ from the left. Then, we separate v into two parts at the position, making a new vertex $v'' \in V_m$ such that $v''[0, m - 1]$ is the right half of $v[0, m - 1]$. The left half is identified with v' (see Figure 4). We note that

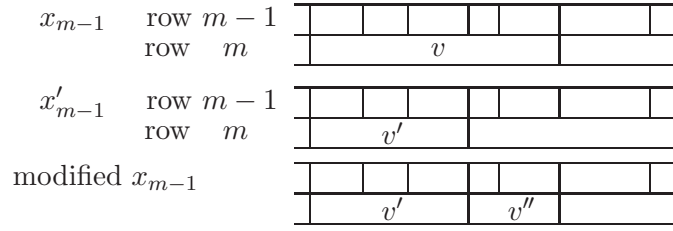


FIGURE 4. Change of an m -symbol. There is a common cut at the left end.

v'' is a “new” m -symbol, even if there has been an original symbol v''' with $v'''[0, m-1] = v''[0, m-1]$. We replace every occurrence of the m -symbol v in every element of X_m by the concatenation $v'v''$. In the Bratteli diagram, we delete v and add v'' , and we replace the edges that connect v with the edges that connect v' or v'' . In accordance with the new $(m+1)$ -symbols, each edge in E_{m+1} that had connected v has to be duplicated into two edges, one connecting v' and the other connecting v'' . Further, the orders of the edges of E_{m+1} have to be changed. After all the necessary changes, we show that the new ordered Bratteli diagram has the Bratteli–Vershik system that is canonically isomorphic to the original. We note that the maximal or minimal paths are joined to the infinite cuts from level 0 to ∞ . Because the $(m+1)$ th row is not changed, no new infinite cut arises. Therefore, no new infinite maximal path or infinite minimal path arises. Thus, we need not change the Vershik map up to canonical isomorphism; the existence of canonical isomorphism is evident. The number of vertices is not changed. After this modification, no cut that existed is removed. The coincidence of the m -cut from k_0 to the right might be shortened. Nevertheless, the same modification is possible, and finally, we reach the point where we have the same m -cuts from k_0 to the right end. The same argument is valid for the left direction from k_0 . Now, x_{m-1} and x'_{m-1} have the same m -cuts throughout the sequences. Then, we can apply the previous argument. Thus, we get a factor in the m th row that decreases the number of symbols V_m , and (x_{m-2}, x'_{m-2}) is still m -separated. We can now delete the $(m-1)$ th row and continue this process. Finally, the m th row is represented by only one vertex, and the pair (x_{m-K}, x'_{m-K}) is still m -separated and has a common m -cut, which is a contradiction.

Proof for case (2). For an arbitrarily large n , there exists an $m(n) > n$ such that every pair (x_n, x'_n) of depth n has no common $m(n)$ -cut. As described briefly in [DM], by telescoping, we wish to show that the condition of (2) holds for every n . Fix n . Take an $n' > m(n)$ such that every pair

$(x_{n'}, x'_{n'})$ of depth n' has no common $m(n')$ -cut. Then, because $n' > m(n)$, every pair (x_n, x'_n) of depth n has no common n' -cut. Thus, by telescoping (from n' to n), every pair (x_n, x'_n) of depth n has no common $(n+1)$ -cut. Thus, through consecutive application of such telescoping, we get an ordered Bratteli diagram such that for every $n > 0$, every pair (x_n, x'_n) of depth n has no common $(n+1)$ -cut. For each $n > 0$, let (x_n, x'_n) be a pair of depth n that has no common $(n+1)$ -cut. There exists an $N > 0$ such that every pair (x_n, x'_n) of depth $n \geq N$ satisfies $\sup_{i \in \mathbb{Z}} d(f^i(x_n), f^i(x'_n)) \leq \varepsilon_0$. By Lemma 4.3, for any pair (x_n, x'_n) of depth $n \geq N$, the minimal sets of $\omega(x_n)$ and $\omega(x'_n)$ coincide and the minimal sets of $\alpha(x_n)$ and $\alpha(x'_n)$ coincide. First, we claim that $\omega(x_n) \cap \omega(x'_n)$ with $n \geq N$ contains a periodic point only for finitely many $n \geq N$. Suppose that there exists a periodic point y and an infinite set $\mathbf{N} \subset \{n \mid n \geq N\}$ such that $y \in \omega(x_n) \cap \omega(x'_n)$ for all $n \in \mathbf{N}$. Let Z_n be the positions of the n -cuts of y . Then, Z_i ($i \in \mathbf{N}$) is periodic and $Z_n \supseteq Z_m$ for $n < m$. Therefore, there exists an infinite set $\mathbf{N}' \subset \mathbf{N}$ such that for all $n \in \mathbf{N}'$, the n -cuts of y are the same. Let $n \in \mathbf{N}'$. There exists a sequence $k(1) < k(2) < \dots$ with $\lim_{i \rightarrow +\infty} f^{k(i)}(x_n) = y$. Taking a subsequence if necessary, we get $\lim_{i \rightarrow +\infty} f^{k(i)}(x'_n) = y' \in E_{0,\infty}$. Because $x_n[n] = x'_n[n]$, we get $y[n] = y'[n]$. Thus, y and y' have the same n -cuts. Similarly, y and y' have no common $(n+1)$ -cut. Nevertheless, the positions of the $(n+1)$ -cuts of y are the same as the positions of the n -cuts of y , and the $(n+1)$ -cuts of y' have to be a part of the n -cuts of y' and of y . Thus, y, y' have common $(n+1)$ -cuts, which is a contradiction. We have proved the claim. Therefore, there exists an $N' > N$ such that for all $n \geq N'$, $\omega(x_n) \cap \omega(x'_n)$ contains no periodic point.

By telescoping, there exists a minimal set M such that for all $n \geq 1$, $M \subseteq \omega(x_n) \cap \omega(x'_n)$. The remainder of the argument proceeds as that of the Infection lemma of [DM]. Nevertheless, we cannot assume $M = E_{0,\infty}$ in general. We only need to check that this fact does not cause any problem. Let $i_0 > 0$ and $L > 0$ be arbitrarily large integers. Let $j = i_0 + L$. For each $i \in [i_0, j-1]$, the pair (x_i, x'_i) has depth i with no common $(i+1)$ -cuts. Fix a $y_0 \in M$. As in [DM], for each $i \in [i_0, j-1]$, by applying some element τ_i of the enveloping semi-group, we can get a pair $(\tau_i(x_i), \tau_i(x'_i))$ with $\tau_i(x_i) = y_0$. By letting $y_i = \tau_i(x'_i)$ ($i \in [i_0, j-1]$), we get that each (y_0, y_i) is i -compatible with no common $(i+1)$ -cuts. Let $i, i' \in [i_0, j-1]$ satisfy $i < i'$. Then, $y_{i'}[i+1] = y_0[i+1]$, and $y_{i'}$ has no common $(i+1)$ -cut with y_i . Thus, we get finite elements y_0 and y_i ($i \in [i_0, j-1]$). These elements are all i_0 -compatible and pairwise j -separated. Furthermore, they have no common j -cuts pairwise. We have to take care that y_i ($i \in [i_0, j-1]$) need

not be elements of M . Let $M(i) := \pi_i(M)$ for all $i \geq 0$. The next lemma is called the Infection lemma in [DM].

Infection lemma: Suppose that there exist at least $K^{K+1} + 1$ i -compatible points $y_k \in E_{0,\infty}$ ($k \in [1, K^{K+1} + 1]$), which, for some $j > i$, are pairwise j -separated with no common j -cuts. Let $\hat{y} = y_k[i] \in M(i)$ be the common sequence. Then, \hat{y} is eventually periodic.

The proof is factually identical to that of the original. Thus, we omit the proof here. By the Infection lemma, $M(i_0)$ has a periodic orbit. Nevertheless, by the minimality of M , $M(i_0)$ is also minimal. Thus, $M(i_0)$ is a periodic orbit. Because i_0 can be arbitrarily large, it follows that M is an odometer, which is a contradiction. \square

We think that we have settled down the extension process of [DM, Theorem 1] at least once. As an example, we get the following:

Corollary 4.5. *If (X, f) is a finite-rank proximal zero-dimensional system, then it is symbolic.*

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REFERENCES

- [HPS] R. H. Herman, I. F. Putnam, and C. F. Skau, “Ordered Bratteli diagrams, dimension groups and topological dynamics,” *Int. J. Math.*, vol. 03, no. 06, pp. 827–864, Dec 1992. [Online]. Available: <http://dx.doi.org/10.1142/s0129167x92000382>
- [AK] E. Akin and S. Kolyada, “Li–Yorke sensitivity,” *Nonlinearity*, vol. 16, no. 4, pp. 1421–1433, May 2003. [Online]. Available: <http://dx.doi.org/10.1088/0951-7715/16/4/313>
- [D] A. I. Danilenko, “Strong orbit equivalence of locally compact Cantor minimal systems,” *Int. J. Math.*, vol. 12, no. 01, pp. 113–123, Feb 2001. [Online]. Available: <http://dx.doi.org/10.1142/s0129167x0100068x>
- [M] H. Matui, “Topological orbit equivalence of locally compact Cantor minimal systems,” *Ergod. Th. Dynam. Sys.*, vol. 22, no. 06, pp. 1871–1903, Dec 2002. [Online]. Available: <http://dx.doi.org/10.1017/s0143385702000688>
- [DM] T. Downarowicz and A. Maass, “Finite-rank Bratteli–Vershik diagrams are expansive,” *Ergod. Th. Dynam. Sys.*, vol. 28, pp. 739–747, Feb 2008. [Online]. Available: <http://dx.doi.org/10.1017/s0143385707000673>
- [BKM] S. Bezuglyi, J. Kwiatkowski, and K. Medynets, “Aperiodic substitution systems and their Bratteli diagrams,” *Ergod. Th. Dynam. Sys.*, vol. 29, no. 01, pp. 37–72, Oct 2009. [Online]. Available: <http://dx.doi.org/10.1017/s0143385708000230>
- [S] T. Shimomura, “A Bratteli–Vershik representation for all zero-dimensional systems,” *arXiv:1603.03940*, p. submitted, 2016. [Online]. Available: <https://arxiv.org/abs/1603.03940>

NAGOYA UNIVERSITY OF ECONOMICS, UCHIKUBO 61-1, INUYAMA 484-8504, JAPAN

E-mail address: `tkshimo@nagoya-ku.ac.jp`